J. Fluid Mech. (1975), *vol.* 67, *part* 2, *pp.* 289–297

Printed in Great Britain

Boundary-layer growth at a three-dimensional rear stagnation point

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(Received 20 February 1974)

The theory of boundary-layer growth at a rear stagnation point, first presented by Proudman & Johnson, is here extended to cover fully three-dimensional rear stagnation points. Supporting numerical solutions of the full initial-value problem establish the relevance of the inviscid similarity solutions obtained.

1. Introduction

Recently Robins & Howarth (1972) and Howarth (1973) have presented analyses and numerical solutions for the problems of boundary-layer growth at two-dimensional and axisymmetric rear stagnation points. Their analyses follow the spirit of, and are extensions of, the pioneering paper of Proudman & Johnson (1962).

It is natural to inquire whether or not any progress can be made with the problem of a general three-dimensional rear stagnation point. Such front stagnation points have been studied by Howarth (1951) and Davey (1961). A co-ordinate system (x, y, z) is introduced with the stagnation point in question at $(0, 0, 0)$. The co-ordinate normal to the body is z, and x and y are in the directions of the two principal curvatures. In this co-ordinate system, the velocity outside the boundary layer has components

$$ax, \quad by, \quad -(a+b)z \tag{1.1}$$

for some constants a and b. The signs and relative magnitude of a and b determine the nature of the stagnation point. In this paper we speak of a point of attachment if the normal component of the velocity outside the boundary layer is directed towards the wall, that is, if $a+b > 0$. In the opposite case we speak of a point of separation. If a and b have the same sign, the stagnation point is termed a nodal point; otherwise it is a saddle point. This classification is consistent with the more usual one given in terms of the topography of the skin-friction lines at the body.

The object of this paper is to apply the Proudman–Johnson theory, as far as is possible, to the problem of three-dimensional stagnation points of separation, of both nodal-point and saddle-point type.

The spirit of the argument is the same as in the Proudman & Johnson paper; that is, the appropriate idealization of the flow is assumed to be that near a plane wall, with outer boundary condition (1.1), and a similarity solution of the inviscid form of the equations is sought.

Denoting the velocity $\mathbf{u}$ by (u, v, w) and the time by t, we write

$$\left.\begin{aligned} u = axF_\eta(\eta,\tau), \quad v = byG_\eta(\eta,\tau), \quad w = |a|^{\frac{1}{2}}\nu^{\frac{1}{2}}w^*(\eta,\tau), \\ \eta = z|a|^{\frac{1}{2}}/\nu^{\frac{1}{2}}, \quad \tau = |a|\,t \end{aligned}\right\} \tag{1.2}$$

and substitute into the equations of motion and continuity, to obtain

$$w^* = -\left(\frac{a}{|a|}F + \frac{b}{|a|}G\right), \tag{1.3a}$$

$$F_{\eta\eta\eta} + \left(\frac{a}{|a|}F + \frac{b}{|a|}G\right)F_{\eta\eta} + \frac{a}{|a|}(1+F_\eta^2) = F_{\eta\tau}, \tag{1.3b}$$

$$G_{\eta\eta\eta} + \left(\frac{a}{|a|}F + \frac{b}{|a|}G\right)G_{\eta\eta} + \frac{b}{|a|}(1-G_\eta^2) = G_{\eta\tau}, \tag{1.3c}$$

$$F(0,\tau) = G(0,\tau) = F_\eta(0,\tau) = G_\eta(0,\tau) = 0 \quad \text{for} \quad \tau \neq 0, \tag{1.3d}$$

$$F(\eta,0) = G(\eta,0) = \eta \quad \text{for} \quad \eta \neq 0, \tag{1.3e}$$

$$F_\eta \to 1, \quad G_\eta \to 1 \quad \text{as} \quad \eta \to \infty. \tag{1.3f}$$

When the body is idealized as a plane wall, these are exact solutions of the Navier–Stokes equations, as in the original two-dimensional case. We shall assume without loss of generality that $a < 0$. In the nodal-point case we shall have $b < 0$; in the saddle-point case we shall have $b > 0$, with $b < |a|$ to ensure a point of separation rather than a point of attachment. If we define $c = b/a$, then we have $-1 < c < 0$ for a saddle point of separation, and for a nodal point of separation it is sufficient to restrict attention to the range $0 < c \leqslant 1$. Note that in the saddle-point case our convention is the opposite of that used by Davey, who had $-1 \leqslant c < 0$ for a saddle point of attachment and $c < -1$ for a saddle point of separation. The reason for this particular choice will become clear later; meanwhile it is hoped that the foregoing remarks will prevent any possible confusion.

With $c = b/a$, and the above conventions, $(1.3b, c)$ become

$$F_{\eta\eta\eta} - (F+cG)F_{\eta\eta} + F_\eta^2 - 1 = F_{\eta\tau}, \tag{1.3b$'$}$$

$$G_{\eta\eta\eta} - (F+cG)G_{\eta\eta} + c(G_\eta^2 - 1) = G_{\eta\tau}. \tag{1.3c$'$}$$

We now seek a similarity solution of the inviscid form of $(1.3b, c)'$ of the type

$$\left.\begin{aligned} F(\eta,\tau) = \lambda(\tau)f(\xi), \\ G(\eta,\tau) = \lambda(\tau)g(\xi), \end{aligned}\right\} \quad \xi = \eta/\lambda(\tau), \tag{1.4}$$

and obtain

$$\left.\begin{aligned} -(f+cg)f'' + f'^2 - 1 = (-\dot\lambda/\lambda)\,\xi f'', \\ -(f+cg)g'' + c(g'^2-1) = (-\dot\lambda/\lambda)\,\xi g'', \end{aligned}\right\} \tag{1.5}$$

where a dash or dot denotes a derivative. As in the two-dimensional case, we retain the boundary condition $f(0) = g(0) = 0$, together with the outer boundary condition, which becomes

$$f'(\xi) \to 1, \quad g'(\xi) \to 1 \quad \text{as} \quad \xi \to \infty. \tag{1.6}$$

Separation of the variables gives $\dot\lambda/\lambda = k$, or

$$\lambda = e^{k\tau} \tag{1.7}$$

for some constant of separation k.

We make the same assumption as Proudman & Johnson, that the approach to unity of f' and g' be exponential. Under this condition, straightforward analysis of the asymptotic form of (1.5) gives

$$k = 1+c, \tag{1.8}$$

any other value of k giving at best algebraic decay. Our similarity equations then become

$$\left.\begin{aligned} [f+cg-(1+c)\,\xi]f''+1-f'^2 &= 0,\\ [f+cg-(1+c)\,\xi]\,g''+c(1-g'^2) &= 0,\\ f(0)=g(0)=0,\quad f'(\infty)=g'(\infty) &= 1.\end{aligned}\right\} \tag{1.9}$$

We can immediately obtain a first integral of (1.9) because

$$[f+cg-(1+c)\,\xi] = (f'^2-1)/f'' = c(g'^2-1)/g'' \tag{1.10}$$

and so

$$\left(\frac{1-f'}{1+f'}\right) = K\left(\frac{1-g'}{1+g'}\right)^{1/c}, \tag{1.11}$$

where K is some constant that will depend on c.

2. Saddle-point case

As $\xi \to \infty$, we know that $f' \to 1$ and $g' \to 1$. Hence $(1-f')/(1+f')$ and $(1-g')/(1+g')$ both tend to zero, and further, in the saddle-point case with $c < 0$,

$$\left(\frac{1-g'}{1+g'}\right)^{1/c} \to \infty.$$

This means that (1.11) cannot be satisfied unless either $f' \equiv 1$ and $K \equiv 0$, or $g' \equiv 1$ and $1/K \equiv 0$. Both of these are possible solutions of (1.9). This means that, for the saddle-point case at least, reverse flow does not occur in the inviscid flow region in one or other of the velocity components. It seems clear on physical grounds that reverse flow will not occur in whichever component has its external flow parameter (a or b) positive. Indeed, this is confirmed in the numerical solutions described subsequently. We have agreed on the convention $a < 0$, $b > 0$, and so

$$g' \equiv 1. \tag{2.1}$$

Since $g(0) = 0$, we have immediately

$$g(\xi) = \xi, \tag{2.2}$$

and our similarity equations (1.9) become

$$\left.\begin{aligned} (f-\xi)f''+1-f'^2 &= 0,\\ g &= \xi,\\ f(0)=0,\quad f'(\infty) &= 1.\end{aligned}\right\} \tag{2.3}$$

The solution of (2.3) for f is formally identical to the original two-dimensional Proudman & Johnson solution, viz.

$$f = \xi-(2/d)\,(1-e^{-d\xi}), \quad g = \xi \tag{2.4}$$

for some constant d. The reverse-flow region grows like $\exp\{-(1+c)\tau\}$. This is in agreement with the two-dimensional case when $c = 0$, as expected. The growth is slower as $c \to -1$. The case $c = -1$ is a limiting case, and exhibits no reverse flow.

The solution (2.4) can be regarded as the first outer solution in an asymptotic series. It is not valid in the vicinity of the body, where the viscous forces are important. However, we can now examine the flow in the immediate neighbourhood of the body. For small ξ, we have

$$f \sim -\xi, \quad g \sim \xi,$$

which gives

$$F \sim -\eta, \quad G \sim \eta.$$

Bearing in mind that $a < 0$ and $b > 0$, this indicates that the flow near the wall is that for a *nodal point of attachment*. Indeed, writing

$$F = -f_0(\eta), \quad G = g_0(\eta), \quad c^* = -c \tag{2.5}$$

for our first inner solution, we obtain

$$f_0''' + (f_0 + c^* g_0) f_0'' + 1 - f_0'^2 = 0, \tag{2.6a}$$

$$g_0''' + (f_0 + c^* g_0) g_0'' + c^*(1 - g_0'^2) = 0, \tag{2.6b}$$

$$f_0(0) = f_0'(0) = g_0(0) = g_0'(0) = 0, \tag{2.6c}$$

$$f_0' \to 1, \quad g_0' \to 1 \quad \text{as} \quad \eta \to \infty, \tag{2.6d}$$

$$c^* > 0, \tag{2.6e}$$

which is precisely Howarth's (1951) problem for a nodal point of attachment. It is of course necessary to verify that (2.4) is the relevant asymptotic solution to our problem. To do this we appeal to a numerical integration of the full problem, and adopt a procedure identical to that of Proudman & Johnson, that is, plot $\log(1-f')$ against ξ, and expect a straight line (except very close to the body), from the slope of which the constant d can be found. This has been done (although to save space the graphs are not reproduced here) for $c = -0{\cdot}05$ and $c = -0{\cdot}5$. The expected straight line does indeed materialize, and the value of the constant d that emerges is $3{\cdot}53$ when $c = -0{\cdot}05$ (close to the $c = 0$ result of $3{\cdot}51$) and $6{\cdot}15$ when $c = -0{\cdot}5$. Also, the values for the x and y components of the skin friction obtained from the numerical solutions clearly tend towards the values expected assuming a nodal point of attachment at the wall. The first-order saddle-point problem may thus be considered solved.

3. Nodal-point case

In the saddle-point case of the last section, it was discovered that reverse flow occurs in one velocity component only. In the nodal-point case we must expect different behaviour, at least for certain values of c. For $c = 1$ we must recover the axisymmetric results of Howarth (1973), with $f = g$, and hence reverse flow in both components. The question is, does this type of behaviour continue right down to $c = 0$, or is the behaviour for small positive c more akin to the saddle-point case, with reverse flow in one component only? This latter possibility would

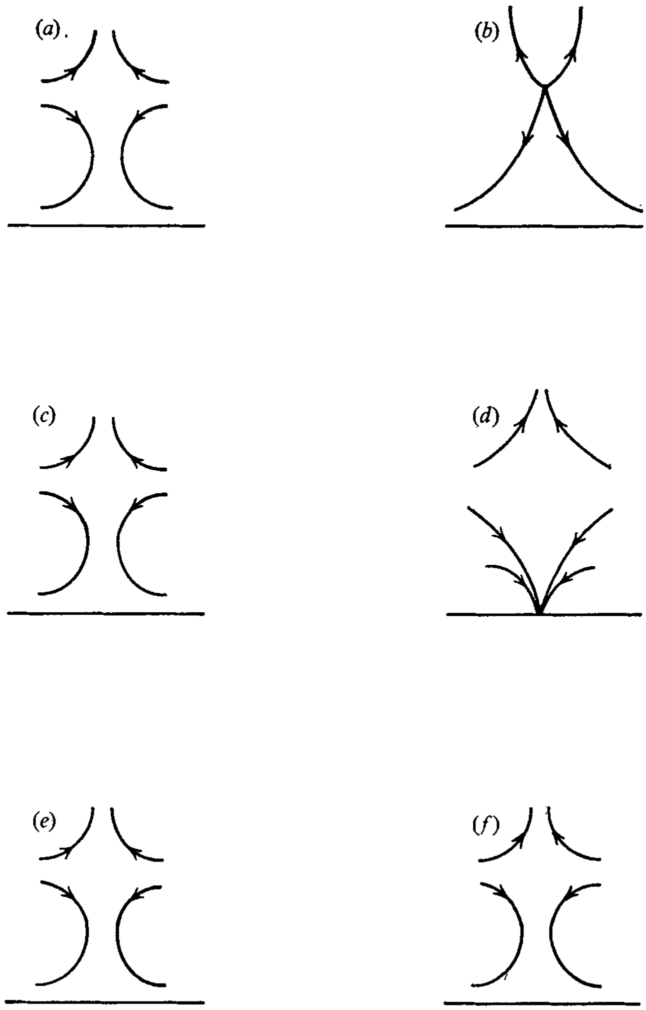

FIGURE 1. Sketch (representative) of streamlines in the x, z and y, z planes as follows: (*a*) saddle point, x, z plane; (*b*) saddle point, y, z plane; (*c*) nodal point, $c < c_0$, x, z plane; (*d*) nodal point, $c < c_0$, y, z plane; (*e*) nodal point, $c > c_0$, x, z plane; (*f*) nodal point, $c > c_0$, y, z plane.

imply the existence of a critical value of c, say c_0, in $(0, 1)$, where the character of the inviscid flow changes. If $c < c_0$, there would be reverse flow in one component only, and if $c > c_0$, there would be reverse flow in both components. The numerical evidence (which will be described later) strongly supports this view. Physically, it appears that, if c is small and positive, the curvature in the y direction is not very large and the x component of the flow is able to bleed off enough fluid to prevent flow reversal in the y component occurring at all. The precise values of c_0 will be discussed later. For the present we assume its existence on numerical evidence. The supposed flow configuration then implies that the stagnation point of attachment close to the wall is of saddle-point type for $c < c_0$ and nodal-point type for $c > c_0$. Now let us examine these two cases in greater detail.

Case $c < c_0$

The inviscid similarity solution is here very similar to the saddle-point case, that is

$$f = \xi - (2/d)(1 - e^{-d\xi}), \quad g = \xi \tag{3.1}$$

for some constant d. Near the wall we have, for the inner viscous region,

$$F \sim -\eta, \quad G \sim \eta,$$

and if we write

$$F = -f_0(\eta), \quad G = g_0(\eta), \quad c^* = -c,$$

then we again obtain (2.6), except that this time $c^* < 0$, and so we have a Davey type of *saddle point of attachment* at the wall. Note that we can again verify that (3.1) is the correct asymptotic representation of the outer flow, by doing a full numerical integration and plotting $\log(1-f')$ against ξ. This has been done for the case $c = 0{\cdot}1$. The expected straight line again materializes, and the value of the constant d is $3{\cdot}57$ for this case. The x and y components of the skin friction also clearly tend to the values expected for a saddle point of attachment near the wall, thus offering further confirmatory evidence. Hence for $c < c_0$ (whatever c_0 may be; certainly it is greater than $0{\cdot}1$!) the first-order nodal-point problem may be considered solved.

Case $c > c_0$

This case is more difficult, since both velocity components exhibit reverse flow, and our similarity equations are given by (1.9) without any simplifying features. The author has not been able to find an analytical solution to this pair of coupled nonlinear ordinary differential equations, except for the case $c = 1$, $f = g$. However, there is little doubt that they do correctly describe the first approximation to the outer flow, for reasons which will be given later. Note that the system (1.9) and the associated boundary conditions are invariant under the transformation

$$f \to \alpha f, \quad g \to \alpha g, \quad \xi \to \alpha \xi. \tag{3.2}$$

Thus we have a choice of a positive constant α at our disposal when comparing the similarity solution with any numerical integration of the full problem. This constant plays precisely the same role as the constant c in the original Proudman & Johnson paper, reflecting an uncertainty in the asymptotic similarity solution as to the precise location of the time origin. In this context it may be helpful to think of α as a scale stretching constant at our disposal to make the similarity solution fit the numerical solution as well as possible. Since (1.9) cannot be solved analytically, two choices are open to us. Either solve it numerically for a particular value of c, and compare such results with the full numerical integration, or else attempt to find perturbation solutions about $c = 1$. Both of these courses have been followed. Figure 2 shows the numerical similarity solution for $c = 0{\cdot}75$, except for the region close to the wall, where it is not relevant. The numerical solution of the full initial-value problem cannot be drawn separately from this on a graph, and so, for the values of ξ chosen, may be regarded as the same curve. Clearly the agreement is excellent. Furthermore the skin-friction components do

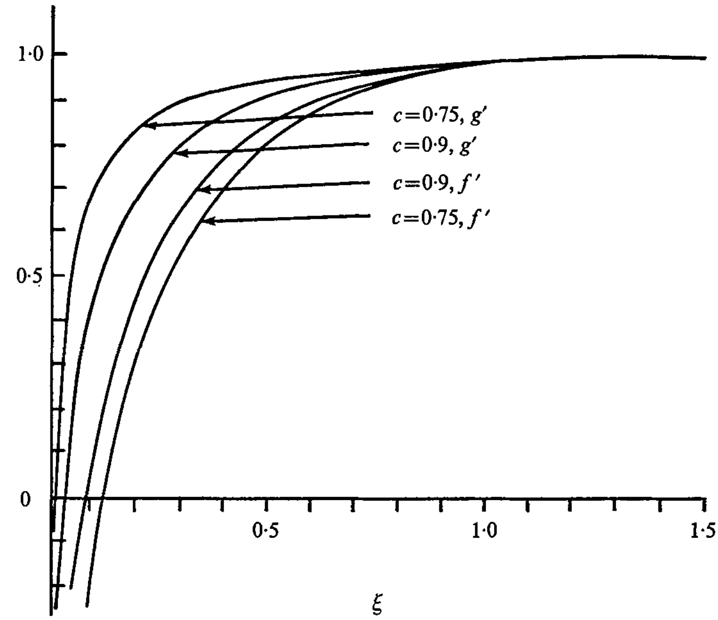

FIGURE 2. Similarity solutions for $c = 0{\cdot}75$ and approximate similarity solutions for $c = 0{\cdot}9$, both coincident with numerical solutions for values plotted.

indeed approach those predicted by assuming a nodal point of attachment in the viscous inner layer near the wall.

Near $c = 1$, an approximate solution of the similarity equations can be obtained by writing (1.9) in the form

$$\left.\begin{aligned} f''/(f'^2-1) &= 1/[f+cg-(1+c)\,\xi],\\ g''/(g'^2-1) &= c/[f+cg-(1+c)\,\xi] \end{aligned}\right\} \tag{3.3}$$

and iterating, using the axisymmetric solution as a starting point. This gives

$$\left.\begin{aligned} \left(\frac{1-f'}{1+f'}\right) &= \gamma\left(\frac{1-f_0'}{1+f_0'}\right)^{2/(1+c)},\\ \left(\frac{1-g'}{1+g'}\right) &= \delta\left(\frac{1-f_0'}{1+f_0'}\right)^{2c/(1+c)} \end{aligned}\right\} \tag{3.4}$$

for some constants γ and δ, where f_0 is the axisymmetric solution. The role of the constants γ and δ is not clear, though presumably these too must be chosen to give the best fit between the numerical and the similarity solutions. Also it is a little strange that there are two such constants, in view of (3.2). However, in principle they are related by (1.11), and any choice of these must be consistent with the value of K obtained for any particular c. The comparison has been made for the case $c = 0{\cdot}9$, and the results are displayed in figure 2. Again the agreement is excellent, the numerical and similarity curves being indistinguishable on a graph, for the values of ξ plotted (i.e. excepting points very close to the wall). It was found that $\gamma = 1{\cdot}6$ and $\delta = 0{\cdot}67$ for this value of c. For $c = 1$ the values

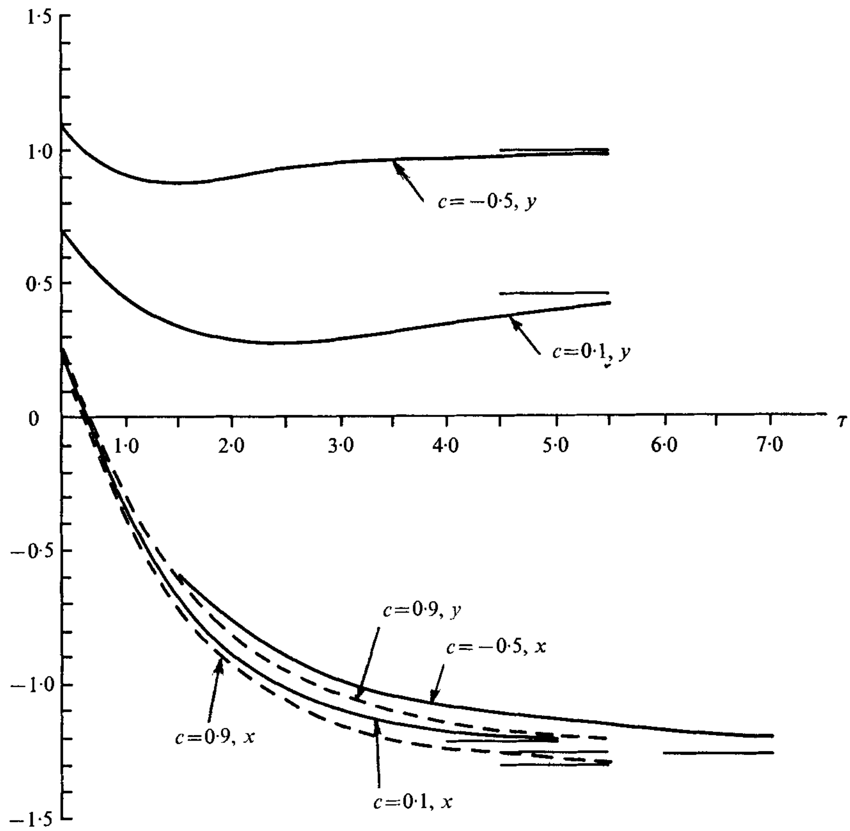

FIGURE 3. Curves of x and y components of skin friction *vs*. time. The straight lines are the expected asymptotic limits.

would of course be $\gamma = \delta = 1$, and at first sight this is a large variation; however the behaviour of the solutions as a function of c was generally found to be very nonlinear. An alternative method of approximation is to develop a series in powers of $1-c$ about the $c = 1$ solution. This has been done, but the approximations obtained are not quite as good as those obtained from the above results, and so are not presented.

4. Higher-order approximations for the saddle-point case

It is possible to make some progress towards higher-order approximations for the saddle-point case, in fact as far as the second outer and second inner problem, following the methods used by Robins & Howarth. The details are not presented here, but the interested reader is referred to Howarth (1972) for further details of this, and other aspects of the work described herein.

5. Numerical solutions

The precise method of obtaining numerical solutions to the full viscous initial-value problem is the same in principle as that used by Robins & Howarth, and so a description will not be given here. Results have been obtained for the cases $c = -0{\cdot}5,\ -0{\cdot}05,\ 0,\ 0{\cdot}25,\ 0{\cdot}3,\ 0{\cdot}4294,\ 0{\cdot}5,\ 0{\cdot}75,\ 0{\cdot}9$ and 1. Sample graphs of the x and y components of the skin friction *vs.* time are presented in figure 3, for $c = -0{\cdot}5$, $0{\cdot}1$ and $0{\cdot}9$. The numerical results indicate that the similarity solution is definitely the correct asymptotic solution to the problem. The only outstanding question is that of the precise value of c_0, where the character of the flow changes for c positive. The numerical results indicate that, for $c = 0{\cdot}1$ and $0{\cdot}3$, the skin-friction curves definitely tend towards the values expected for a saddle point of attachment near the wall. For $c = 0{\cdot}9$, $0{\cdot}75$ and $0{\cdot}5$, a nodal point of attachment is just as clearly indicated. Near $c = 0{\cdot}4$ the skin-friction curves seem to change only slowly with time (at least, after the initial stages of setting up reverse flow are beginning to settle down), and since for technical reasons the numerical integration had to terminate at $t = 5{\cdot}5$ for these values of c, it is a little difficult to see precisely what is going on. Now c_0 definitely lies between $0{\cdot}3$ and $0{\cdot}5$, and is probably near $0{\cdot}4$. It seems very possible that the critical value found by Davey ($0{\cdot}4294$, allowing for a sign change) is again a critical value for these separating flows; and just as Davey expected that the outer flow (and therefore the inner flow) would change drastically from $c < c_0$ to $c > c_0$, so the same happens here.

The author is deeply indebted to Dr W. H. H. Banks, of the University of Bristol, for much help and encouragement. Thanks are also due to the Science Research Council for financial support.

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